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Non-relativistic abstract continuum mechanics and its possible physical interpretations

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Abstract
Non-relativistic causal mechanics of abstract continuum with measure is developed and its possible interpretations are suggested. The theory is based on two main principles: the fundamental causality principle which gives the measure conservation law, and the so-called basic principle of dynamics which is usually written in the form of the momentum balance equation. Being based on the measure conservation law, the theory is non-relativistic. Two levels of the theory are presented. Each level results in a non-closed system of equations. This requires additional external (physical) considerations which are also discussed. The presence of two levels of the theory reflects the existence of two main physical interpretations—hydromechanical and electromagnetic. Their connections with the general theory and its particular cases are considered.

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1. Introduction

Causal mechanics of abstract continuum with measure and its possible interpretations are considered. A set of images of diffeomorphic mappings of the body (abstract continuum) into a metric space with pseudo-Riemann metrics is studied. The theory is based on two principles. The first one is the fundamental *causality principle* which gives the *measure conservation law* (or the *measure balance equation*). The second one is the so-called *basic principle of dynamics* which is usually written in the form of the *equation of motion* (or the *momentum balance equation*). Additional closing relations are also postulated. Being based on the measure conservation law, the theory under consideration is non-relativistic.

Two levels of the theory are presented. In the general case, the second level follows from the first one, but not vice versa. Each level results in a non-closed system of equations which requires additional external (physical) considerations.

Table 1. Hydromechanical analogues of electromagnetic quantities. Here, M, C, L and T in subcolumns are mass, charge, length and time scales, correspondingly.

Electromagnetism			Hydromechanics		
Vector potential,	$\vec{A}$	$\frac{M}{C} \cdot \frac{L}{T}$	3D velocity,	$\vec{u}$	$\frac{L}{T}$
Magnetic induction,	$\vec{B}$	$\frac{M}{C} \cdot \frac{1}{T}$	Curl,	$\vec{\omega}$	$\frac{1}{T}$
Electric field strength,	$\vec{E}$	$\frac{M}{C} \cdot \frac{L}{T^2}$	Lamb vector,	$\vec{l}$	$\frac{L}{T^2}$
Electric potential,	φ	$\frac{M}{C} \cdot \frac{L^2}{T^2}$	Bernoulli integral,	$\mathcal{H}$	$\frac{L^2}{T^2}$

The presence of two levels of the theory reflects the existence of two main physical interpretations. They will be called hydromechanical and electromagnetic. Most often each interpretation is bound with one of the levels suggested. As a rule, the hydromechanical interpretation is used in the case of the first level theory, whereas the electromagnetic interpretation is usually connected with the second level. However, exceptions exist. For instance, the vorticity equation, which is considered here as a part of the second level theory, is widely used in the fluid mechanics and has its twin in the electromagnetic field (EMF) theory known as the Faraday's law (see section 2.2.1). Similarly, in the case of the electromagnetic interpretation the hydrodynamical Maxwell's equations are sometimes considered (see, e.g., [1]).

Hydromechanical interpretation of the general 4D first level theory has been studied in [2]. Classical fluid mechanics being a particular 3D case of the general theory was discussed in [3]. Electromagnetic interpretation of the current theory is presented here for the first time.

A remarkable similarity between the Maxwell's electromagnetic field equations and the hydromechanical equations of the incompressible perfect fluid is well known. Indeed, the analogy between the relations binding the magnetic induction $\vec{B}$ and the electric field strength $\vec{E}$ with the potentials $\vec{A}$ and φ in the EMF theory on the one hand

$$\vec{B} = \nabla \times \vec{A}, \tag{1}$$

$$\vec{E} = -\partial_t \vec{A} - \nabla\varphi, \tag{2}$$

and the definition of vorticity $\vec{\omega}$ together with the equation of motion in the so-called Gromeka–Lamb form (see, e.g., [4]) in the fluid mechanics on the other hand

$$\vec{\omega} = \nabla \times \vec{u}, \tag{3}$$

$$\vec{l} = -\partial_t \vec{u} - \nabla\mathcal{H}, \tag{4}$$

is striking. Here, $\vec{l} \equiv \vec{\omega} \times \vec{u}$ is the Lamb vector and $\mathcal{H}$ is the Bernoulli integral.

Physical dimensions of the EMF and hydromechanical quantities (see table 1 where M, C, L and T are the mass, charge, length and time scales, correspondingly) also demonstrate the above-mentioned similarity. They coincide to within one and the same normalizing factor with the dimension of $\frac{[\text{mass}]}{[\text{charge}]} = \frac{M}{C}$.

The existence of such evident coincidences seems not to be accidental and needs to be explained. From time to time, publications appear considering the analogies between both theories. Some authors use them to find out the physical meaning of the EMF theory (see, e.g., [5–8]). Others are seeking for new perspectives in adjacent fields of knowledge [9, 10]. However, irrespective of all particular goals it is possible to ask a general question: Why these analogies between two fields of knowledge are possible? The current work is an attempt to suggest an answer. We try to show that the Maxwell's equations and the classical fluid mechanics equations are two different physical interpretations of particular cases of the non-relativistic causal abstract continuum mechanics (ACM).

Contrary to some authors which treat field quantities as fundamental entities whereas measures (e.g., charges and currents) as byproducts of the former we follow an opposite view. The thing is that since a physical theory is a description method of experimental data its verification requires measurements and observations. Each measurement assumes the presence of measure (mass, charge, etc). In this sense we regard measures as primary, fundamental notions. They are the basis of our considerations. The field quantities being defined locally cannot be measured and always need to be calculated. In this sense they act as secondary notions.

The paper is organized as follows. We begin with the development of the abstract continuum mechanics (section 2). Two levels of the theory are presented. Section 3 is devoted to a discussion of possible physical interpretations and some particular cases which are directly connected with existing classical theories. The last section 4 contains some concluding remarks.

2. Abstract continuum mechanics

2.1. First level theory

2.1.1. Time and configuration. A body $\mathcal{B}$ will denote a set in a topological space. This means that it will be considered as an infinitely divisible set of points every two of which have non-overlapping vicinities. A set of images $\mathcal{B}_t$ of diffeomorphic mappings of the body $\mathcal{B}$ into a space $\mathcal{W}$ is studied. Each image is numbered with a real parameter t which will be called *time*. The space $\mathcal{W}$ and an image $\mathcal{B}_t$ will be called the *space of events* (or *the spacetime continuum*) and a *configuration* of the body at time t, respectively.

In physical applications, however, it is convenient to bind these mappings with observations and to describe them in terms of the speed of a signal used for observations (see, e.g., [11]). In this case, the space of events $\mathcal{W}$ is considered as a congruence of one-dimensional curves called *world lines* and each point of the body is associated with its *world line*. The world lines of the points of the body in total form the *world tube* of the body $\mathcal{B}$ which is regarded to be a four-dimensional manifold $\mathcal{B}^4$ in the space $\mathcal{W}$.

Each world line may be smoothly parameterized with a real parameter t. Parameterization is arbitrary and two additional requirements may be used to narrow the choice.

(1) The first general requirement is connected with a number of independent parameters. The goal is to replace infinite independent world-line parameters by a single one. One of the world lines is marked out and its parameterization is left arbitrary. For convenience it will be called *the world line of the observer* and its parameter will be called *time*. All other parameters are synchronized with the observer's time.
In applications the synchronization method is usually chosen such that it may be interpreted in terms of the speed of a signal used for observations. The phase speed of the signal will be further denoted by c.

(2) The second requirement concerns the tangent vector field defined on the world tube. A smooth parameterization induces a tangent vector field $\vec{\tau} = d_t$ on the manifold $\mathcal{B}^4$. It is convenient to choose the time such that the length of any vector $\vec{\tau}$ tangent to a world line (velocity vector) is constant, $|\vec{\tau}| = \text{const}$.

The synchronization allows definition of spaces of synchronous events $\mathcal{W}_t$. The space of events $\mathcal{W}$ itself may be considered as a foliation of spaces $\mathcal{W}_t$ the choice of which defines the observer, i.e. the appointed world line and its parameterization (or the signal speed). A synchronous cut $\mathcal{B}_t$ of the world tube $\mathcal{B}^4$ (i.e. an image of the body $\mathcal{B}$) is called a *configuration* of the body at time t. Different observers induce different spaces of synchronous events (or in other words, different foliations) and possess different configurations of the body for the same time value.

2.1.2. Measures and conservation laws. Next, we define a measure $\mathcal{V}(\mathcal{W})$ on the space of events $\mathcal{W}$, which induce measure $\mathcal{V}(\mathcal{W}_t)$ on the spaces of synchronous events $\mathcal{W}_t$, and a measure $\mathcal{M}(\mathcal{B})$ on the body $\mathcal{B}$. These measures ($\mathcal{M}(\mathcal{B})$ and $\mathcal{V}(\mathcal{W}_t)$) induce measures $m(\mathcal{B}_t)$ and $V(\mathcal{B}_t)$, respectively, on the world tube cut $\mathcal{B}_t$. The first measure $m(\mathcal{B}_t)$ being induced by the measure $\mathcal{M}(\mathcal{B})$ is time independent and the *measure m conservation law*

$$d_t m(\mathcal{B}_t) = 0 \tag{5}$$

holds.

The second measure $V(\mathcal{B}_t)$ is in general time dependent. For convenience we shall call it the *volume* of the configuration $\mathcal{B}_t$. The measure m is assumed to be absolutely continuous with respect to the volume V, i.e. $V(\mathcal{P}) = 0 \Leftrightarrow m(\mathcal{P}) = 0$ for any configuration of arbitrary body $\mathcal{P}$. In this case, there exists a unique function ρ defined on $\mathcal{B}_t$ such that

$$m(\mathcal{B}_t) = \int_{V(\mathcal{B}_t)} \rho \, \mathrm{d}V. \tag{6}$$

This is due to the known Radon–Nikodym theorem. The function ρ is called the *density of the measure m* (the derivative of the measure m with respect to the volume V) or the *m-density* for brevity.

The integral conservation law (5) leads to a differential equation known as *the continuity equation for the measure m*:

$$\int_V \mathrm{div}(\rho\vec{\tau}) \, \mathrm{d}V = 0 \quad \Rightarrow \quad \mathrm{div}(\rho\vec{\tau}) = 0. \tag{7}$$

2.1.3. Energy. Together with the measure $\mathcal{M}(\mathcal{B})$ it is possible to consider another measure $C\mathcal{M}(\mathcal{B})$, $C = \text{const}$, on $\mathcal{B}$ and, thus, to induce a measure $Cm(\mathcal{B}_t)$ on the configuration $\mathcal{B}_t$. Its density will be denoted by κ

$$Cm(\mathcal{B}_t) = \int_{V(\mathcal{B}_t)} \kappa \, \mathrm{d}V. \tag{8}$$

Since this measure is time independent also the measure $Cm(\mathcal{B}_t)$ *conservation law*

$$d_t Cm(\mathcal{B}_t) = 0 \tag{9}$$

holds. Corresponding differential equation reads

$$\mathrm{div}(\kappa\vec{\tau}) = 0. \tag{10}$$

In the case $\kappa = \frac{1}{2}|\vec{\tau}|^2\rho$, the measure $Cm(\mathcal{B}_t)$ is called *the total energy* of the configuration $\mathcal{B}_t$ and equations (9) and (10) are called *the total energy conservation laws* (integral and differential, respectively).

Due to physical reasons, the so-called *thermodynamic case* is usually studied. The total energy is endowed with additional structure via substitution of the congruence of the world lines of the points of the body for a congruence of averaged (smoothed) world lines. The averaging operator is a continuous linear projector mapping a set of non-smoothed objects onto a set of averaged objects [12]. Introduction of the averaging procedure (see [13] for discussion concerning this procedure) divides world lines into two parts: smooth averaged curves and pulse curves. The corresponding tangent vector field $\vec{\tau}$ also divides into an averaged vector field $\vec{v}$ which vectors are tangent to the smoothed world lines, and a pulse vector field $\vec{\tau}'$

$$\vec{\tau} = \vec{v} + \vec{\tau}'.$$

In its turn, the total energy divides into the so-called *kinetic energy* $K(\mathcal{B}_t)$ of the configuration of the body and its *internal energy* $E(\mathcal{B}_t)$

$$Cm(\mathcal{B}_t) = K(\mathcal{B}_t) + E(\mathcal{B}_t), \qquad K, E > 0$$

with densities $k \equiv \frac{1}{2}\rho|\vec{v}|^2$ and $e = \frac{1}{2}\rho(|\vec{\tau}|^2 - |\vec{v}|^2)$, respectively.

In the thermodynamic case, equation (5) does not change and equation (8) turns into the total energy $Cm = K + E$ conservation law

$$d_t(K + E) = d_t \int_V (k + e)\,\mathrm{d}V = 0$$

and

$$\mathrm{div}(k + e)\vec{v} = \rho d_t \frac{k + e}{\rho} = 0. \tag{11}$$

2.1.4. Co-ordinate system. Taking into account the existence of the preferential world line of the observer, it is convenient to define a co-ordinate system such that this world line coincides with one of the axes. The space of events is assumed to be homogeneous, thus the origin of the co-ordinate system may be chosen at an arbitrary point of the observer's world line.

The map $\phi^t : \mathcal{W} \to \mathrm{i}\mathbb{R}^1 \times \mathbb{R}^3$, where $\mathrm{i}\mathbb{R}^1$ denotes the space of imaginary numbers, is called the *observer's frame of reference*. It equips each point $P \in \mathcal{W}$ with four numbers $x_P = (x^0, x^1, x^2, x^3)$, the co-ordinates of the point. World line of a body point is defined by four functions $x^0(t), x^1(t), x^2(t), x^3(t)$ which give (Euler) co-ordinates of a point for each time value. It is assumed that $\mathrm{d}x^0 = \mathrm{i}c\,\mathrm{d}t$, where $\mathrm{i} = \sqrt{-1}$ and c is the signal velocity.

Components of vectors will be further considered with respect to the co-ordinate basis $\{\vec{e}_\alpha\}_{\alpha=0}^3$, $\vec{e}_\alpha = \partial_{x^\alpha}$ chosen at each point of $\mathcal{W}$. The index summation convention will be used. From now on the Latin indices will take values from 1 to 3 and Greek indices will take values from 0 to 3.

2.1.5. Metric tensor

Non-thermodynamic case. To express density κ in terms of velocity components, one needs the introduction of a metric tensor, which is a real linear symmetric two-argument function $\mathsf{g}(\vec{v}, \vec{w})$ defined on vectors. Using the standard notation the length of the velocity vector and the total energy density are expressed by definition as follows:

$$|\vec{\tau}|^2 = \mathsf{g}_{\alpha\beta}\tau^\alpha\tau^\beta, \qquad \kappa = \tfrac{1}{2}\rho\mathsf{g}_{\alpha\beta}\tau^\alpha\tau^\beta. \tag{12}$$

Components of the metric tensor depend on the choice of the basis vectors, and in the case of an orthogonal basis the tensor g is diagonal $|\vec{\tau}|^2 = \sum_{\alpha=0}^3 \mathsf{g}_{\alpha\alpha}\tau^\alpha\tau^\alpha$. The values $\mathsf{g}_{\alpha\alpha}$ may be interpreted as scaling factors. In order to define them uniquely one may set $\mathsf{g}_{\alpha\alpha} = g_0$ for all α, and regard $|\vec{\tau}|^2 = 1$ for each tangent vector. Then

$$g_0 = \frac{1}{\sum_\alpha \tau^\alpha\tau^\alpha}. \tag{13}$$

Thus, the metric tensor is proportional to the identity tensor I:

$$\mathsf{g} = g_0\mathsf{I} \quad\Rightarrow\quad \mathrm{d}t^2 = g_0 \sum_\alpha (\mathrm{d}x^\alpha)^2 = g_0(-c^2\,\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2).$$

In this case, the total energy density κ is equal numerically to $\frac{1}{2}\rho$.

Thermodynamic case. Consider the length of velocity vector $1 = |\vec{\tau}|^2 = |\vec{v}+\vec{\tau}'|^2$. Due to the definition of the kinetic energy density $k = \frac{1}{2}\rho|\vec{v}|^2$, the internal energy density is as follows: $e = \frac{1}{2}\rho(1 - |\vec{v}|^2)$. The introduction of the energy structure leads to the redefinition of the metric coefficient, which now is written as (cf formula (13))

$$g_0 = \frac{1 - 2e/\rho}{\sum_\alpha v^\alpha v^\alpha}. \tag{14}$$

2.1.6. Momentum balance. A $\binom{1}{1}$ tensor $\mathsf{M} = \rho\vec{v} \otimes \tilde{v}$ is called a 4-*momentum flux density tensor*. The quantity $\tilde{v} = \mathsf{g}(\vec{v})$ is the velocity 1-form. It is possible to rewrite equation (11) in terms of the momentum flux density. Using the continuity equation (7), equation (11) may be written as follows:

$$\begin{aligned} \operatorname{div}(k+e)\vec{v} &= (\operatorname{div}\mathsf{M})(\vec{v}) + \rho d_t\frac{e}{\rho} \\ &= \left(\operatorname{div}\mathsf{M} + \rho \mathrm{d}\frac{e}{\rho}\right)(\vec{v}) = 0. \end{aligned} \tag{15}$$

Equation

$$\operatorname{div}\mathsf{M} = \operatorname{div}\mathsf{T}, \tag{16}$$

where T is a so-called *stress tensor*, is postulated. Equation (16) is known as the *momentum balance equation* or the *equation of motion*. Taking into account the continuity equation, the equation of motion may also be written in the form

$$\rho d_t\tilde{v} = \operatorname{div}\mathsf{T} \tag{17}$$

which will be used further. Combining (16) and (15), one obtains the *internal energy balance equation*

$$\rho d_t\frac{e}{\rho} = -(\operatorname{div}\mathsf{T})(\vec{v}). \tag{18}$$

Equations (7), (16) and (18) together with tensor T appropriately chosen and closed using some kind of additional relation known as *an equation of state*, make the system of equations of *the first level* of the abstract continuum mechanics.

2.1.7. Stress tensor. The internal structure of the 4D tensor T is invented in accordance with its 3D analogue and is based on the physical interpretation used.

In the simplest case, the stress tensor T is set to be proportional to the metric tensor. Since $\binom{1}{1}$ tensors are considered, this proportionality means

$$\mathsf{T} = \pi\mathsf{I},$$

where I is the unit tensor. Geometrical interpretation of π as a scalar curvature of the spacetime is discussed in [3]. The scalar quantity $p \equiv -\pi g_0^{-1}$ is a function of co-ordinates and is called *pressure*. Continuum model which includes this definition of the stress tensor is known as perfect or ideal.

Another variant of the stress tensor includes a correction term dependent on the symmetric part of the velocity gradient $\nabla\vec{v}$, i.e. the tensor D which is called the *deformation rate tensor*

$$\mathsf{T} = \pi\mathsf{I} + 2\eta\mathsf{D}. \tag{19}$$

The components of these tensors are as follows:

$$2\mathsf{D}_\alpha^\beta = \mathsf{g}^{\beta\gamma}(v_{\gamma;\alpha} + v_{\alpha;\gamma}), \qquad \mathsf{T}_\alpha^\beta = \pi\delta_\alpha^\beta + \mu\delta^{\beta\gamma}(v_{\gamma;\alpha} + v_{\alpha;\gamma}). \tag{20}$$

Here, $(\cdot)_{;\beta}$ denotes the covariant derivative in the direction of $\vec{e}_\beta$, $\delta^{\alpha\beta}$ and δ_α^β are the components of the identity tensor I and $\mu \equiv \eta g_0^{-1}$ is the *dynamic viscosity coefficient* which is frequently considered as constant. This continuum model is called viscous.

2.2. Second level theory

Derivation of the ACM equations of the second level requires consideration of the so-called vorticity equation as an intermediate step. Together with the continuity equation it may be regarded as a bridge between both levels of the theory.

2.2.1. Vorticity equation. First, we rewrite the acceleration 1-form

$$d_t v_\alpha = v^\beta v_{\alpha;\beta} = 2v^\beta \mathsf{w}_{\alpha\beta} + v^\beta v_{\beta;\alpha} = \ell_\alpha + \left(\frac{k}{\rho}\right)_{,\alpha} \quad \Rightarrow \quad d_t\tilde{v} = \tilde{\ell} + \mathrm{d}\frac{k}{\rho}. \tag{21}$$

Here, $(\cdot)_{,\beta}$ denotes the partial derivative in the direction of $\vec{e}_\beta$. The quantity $\tilde{\ell} \equiv 2\tilde{\mathsf{w}}(\vec{v})$ is the 4D 1-form which will be called the *Lamb 1-form* by analogy with the known 3D Lamb vector. Its components are $\ell_\alpha = 2v^\beta \mathsf{w}_{\alpha\beta}$. The quantity w, which components are $\mathsf{w}_{\alpha\beta} \equiv \frac{1}{2}(v_{\alpha;\beta} - v_{\beta;\alpha})$, will be called a *vorticity 2-form.* It is a skew-symmetric part of the velocity gradient, i.e. it is equal to the exterior derivative of the velocity 1-form $\tilde{\mathsf{w}} = \mathrm{d}\tilde{v}$. The exterior derivative $\mathrm{d}\tilde{\mathsf{w}}$ is identical zero

$$\mathrm{d}\tilde{\mathsf{w}} \equiv 0. \tag{22}$$

Taking into account definition (19), the equation of motion (17) reads

$$\tilde{\ell} = \frac{1}{\rho}\mathrm{d}\pi - \mathrm{d}\frac{k}{\rho} + 2\nu \operatorname{div} \mathsf{D}, \qquad \nu \equiv \frac{\mu}{\rho}. \tag{23}$$

Any 1-form may be written as a sum of two terms, one of which is an exact 1-form. In order to isolate the exterior derivative on the right-hand side of (23), we rearrange its last term as follows:

$$2\nu(\operatorname{div}\mathsf{D})_\alpha = \nu\big(v^\beta_{;\alpha\beta} + v_{\alpha;\gamma\beta}\delta^{\beta\gamma}\big) = \nu\big(v^\beta_{;\beta\alpha} - v^\beta \mathsf{R}_{\beta\alpha} + v^\beta_{\alpha;\beta}\big),$$

or in the non-coordinate form

$$2\nu \operatorname{div}\mathsf{D} = \nu(\mathrm{d}(\operatorname{div}\vec{v}) - \mathsf{R}(\vec{v}) + \Box\tilde{v}).$$

The operator $\Box$ is the d'Alembertian and R is the Ricci tensor. Using this expression, one obtains

$$\tilde{\ell} = \mathrm{d}\underbrace{\left(\frac{\pi}{\rho} - \frac{k}{\rho} + \nu \operatorname{div}\vec{v}\right)}_{\equiv\mathcal{H}} + \underbrace{\left(-\pi\,\mathrm{d}\left(\frac{1}{\rho}\right) + \nu(\Box\tilde{v} - \mathsf{R}(\vec{v}))\right)}_{\equiv\tilde{f}} = \mathrm{d}\mathcal{H} + \tilde{f}. \tag{24}$$

By analogy with the 3D hydromechanics the quantity $\mathcal{H}$ will be called here the *Bernoulli integral*. Equation (24) is the 4D variant of the known *Gromeka–Lamb form* of the equation of motion (see, e.g., [4]).

Now due to equation (24) and the identity $\mathrm{d}(\mathrm{d}\mathcal{H}) = 0$, the 2-form $\mathrm{d}\tilde{\ell}$ is equal to

$$\mathrm{d}\tilde{\ell} = \mathrm{d}\tilde{f} \tag{25}$$

which is called the 4D *vorticity equation*. For the incompressible perfect continuum the quantity $\tilde{f} \equiv 0$ and the vorticity equation simplifies to

$$\mathrm{d}\tilde{\ell} = 0$$

or

$$\ell_{\alpha;\beta} - \ell_{\beta;\alpha} = 0.$$

2.2.2. Second level equations. Equations of the second level of the ACM now will be derived.

(1) Definitions of the measure m and m-density remain the same (see equations (5) and (6)).

(2) A 4D vector field $\vec{d}$ is defined such that

$$\operatorname{div}\vec{d} = \rho. \tag{26}$$

This vector field will be called *m-induction*.

(3) Due to the structure of the continuity equation written in terms of differential forms

$$\mathrm{d}(*(\rho\vec{v})) = 0,$$

where $*$ is the Hodge star operator, a 2-form $\tilde{\mathsf{h}}$ may be defined such that locally (see, e.g., [14])

$$\mathrm{d}\tilde{\mathsf{h}} = *(\rho\vec{v}). \tag{27}$$

This 2-form will be called the *vorticity field strength*.

(4) Using representation (24) of the Lamb 1-form $\tilde{\ell} = \mathrm{d}\mathcal{H} + \tilde{f}$, it is possible to define a 1-form $\tilde{\xi} \equiv \tilde{\ell} - \tilde{f}$ such that

$$\mathrm{d}\tilde{\xi} = 0. \tag{28}$$

This equation is just the vorticity equation (25).

(5) Finally, let us consider the energy balance and corresponding second level equation. Using the definitions of $\tilde{\xi}$ and $\mathcal{H}$, one may write

$$\tilde{\xi} = \mathrm{d}\mathcal{H} = \mathrm{d}\frac{\pi}{\rho} - \mathrm{d}\frac{k}{\rho} + \mathrm{d}(\nu\operatorname{div}\vec{v}).$$

This leads to the expression for the gradient of the quantity $\frac{k}{\rho}$

$$\mathrm{d}\frac{k}{\rho} = \mathrm{d}\left(\frac{\pi}{\rho} + \nu\operatorname{div}\vec{v}\right) - \tilde{\xi}.$$

Now, the gradient of the total energy may be written in the form

$$\mathrm{d}\left(\frac{k}{\rho} + \frac{e}{\rho}\right) = \mathrm{d}\left(\frac{\pi}{\rho} + \nu\operatorname{div}\vec{v} + \frac{e}{\rho}\right) - \tilde{\xi} = \mathrm{d}\mathcal{H} - \tilde{\xi}.$$

The latter equality holds due to $\mathrm{d}\frac{e}{\rho} = -\mathrm{d}\frac{k}{\rho}$, which is valid, since $\frac{k+e}{\rho} = 1$.

Acting on the vector $\rho\vec{v}$, the left-hand side of the previous expression gives zero due to the conservation law (11). The right-hand side in its turn gives the equation which is another form of the total energy conservation

$$(\mathrm{d}\mathcal{H} - \tilde{\xi})(\rho\vec{v}) = 0. \tag{29}$$

This equation may be rewritten in a more convenient form. We substitute $\rho\vec{v} = *\mathrm{d}\tilde{\mathsf{h}}$ and write

$$\begin{aligned} 0 &= \varepsilon^{\alpha\beta\gamma\delta}(\mathcal{H}_{,\alpha}\mathsf{h}_{\beta\gamma,\delta} - \xi_\alpha\mathsf{h}_{\beta\gamma,\delta}) \\ &= \varepsilon^{\alpha\beta\gamma\delta}(\underbrace{(\mathcal{H}_{,\alpha} - \xi_\alpha)}_{=0}\mathsf{h}_{\beta\gamma})_{,\delta} - \varepsilon^{\alpha\beta\gamma\delta}(\mathcal{H}_{,\alpha} - \xi_\alpha)_{,\delta}\mathsf{h}_{\beta\gamma} \\ &= \mathrm{d}\xi(*\tilde{\mathsf{h}}), \end{aligned} \tag{30}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is a four-index Levi-Civita symbol. The last equality holds due to $\mathrm{d}(\mathrm{d}(\mathcal{H})) = 0$.

Equations (26)–(28) and (30) make the system of equations of *the second level* of the abstract continuum mechanics. Its connection with the equations of the first level is obvious: equation (27) follows from the continuity equation (7), the equation of motion (16) or its derivatives (17), (23), (24) give birth to equation (28), and equation (30) corresponds to the total energy balance equation (15). The left equation (26) is a definition.

The system contains 12 scalar equations with respect to 19 unknowns $(\rho, \vec{v}, \vec{d}, \tilde{\xi}, \tilde{\mathsf{h}})$. To close the system some additional (material) relations are necessary. Unlike the system of equations (7), (16) and (18) which assumes the existence of the nonzero measure the system (26)–(28) also makes sense in case $\rho = 0$.

3. Interpretations and particular cases

The general case may be interpreted in at least two ways. Both interpretations traditionally deal mainly with one of the two levels presented.

(1) First, one may consider a hydromechanical interpretation and regard measure m as a mass of a physical body. This gives two equivalent 4D descriptions of the physical continuum mechanics. The first level of the ACM (equations (7), (16), (18) and, e.g., (19)) has been suggested in [2] and was called the causal non-relativistic dissipative fluid mechanics. The interpretation of the second level equations consists in the 4D vorticity equation (28) and three other equations (26), (27) and (30) which have never been considered in this context.

(2) Second, in the electromagnetic interpretation the measure m is regarded as the electric charge q of the charged physical body with density ρ_e. Such interpretation of the first level theory gives the continuity equation for charge $\operatorname{div}(\rho_e \vec{\tau}) = 0$ (cf equation (7)) and equation (2). The second level ACM corresponds to the electromagnetic field theory (EMF).

Classical variants of all of these physical theories are three dimensional and may be obtained as interpretations of particular cases of the general ACM theory. Such particular cases include, for example, additional assumptions concerning the signal speed ($c = \infty$ or $c = \text{const}$), viscosity or compressibility. The case $c = \infty$ in the framework of the first level theory corresponds to the classical fluid mechanics. The system of equations of the second level theory together with the assumption $c = \text{const}$ is equivalent to the Maxwell's equations.

3.1. Hydromechanical interpretation

3.1.1. First level theory: non-relativistic causal fluid model. Using the continuity hypothesis a physical body may be thought of as a continuous media and associated with the body $\mathcal{B}$. The measures m and V are interpreted as the mass and volume of the configuration of the body. In this case, the quantity ρ is the mass density and the vector $\vec{v}$ is the velocity of the mean motion of the points of the body. The tensor D is the known *deformation rate tensor*.

The averaging procedure as well as the velocity vector, the internal energy and the viscosity coefficients may be interpreted differently, depending on the spacetime scales of the phenomenon under study. Thus, one may consider either a laminar flow when e and μ have usual physical meaning of the internal energy and the viscosity coefficient, or a turbulent current when e denotes a turbulent energy density and μ is a turbulent viscosity coefficient. In the latter case, the vector $\vec{v}$ is interpreted as a mean velocity of the turbulent flow.

Depending on the choice of the stress tensor T, equations (7), (16) and (18) being closed with an equation of state, make the system of equations of the non-relativistic causal (perfect, viscous or turbulent) fluid mechanics. See [2] for detailed description of the fluid model and [13] for discussion of the averaging procedure.

3.1.2. Second level theory: 4D vorticity equation. General field description of the fluid motion is unusual for the fluid mechanics, though possible. As has been said, equations (26), (27), the 4D vorticity equation (28) and (30) make system of the field equations which describe the non-relativistic dissipative fluid motion. It should be closed with additional relations. One of these relations may be derived using the Newton's gravity law. We consider the equation of motion written in the form (see equation (24) and definition of $\vec{\xi}$)

$$\vec{\xi} = \mathsf{g}^{-1}\,(\mathrm{d}\mathcal{H})\,, \tag{31}$$

and compute the divergence of both sides of it. This gives

$$\operatorname{div}\vec{\xi} = \mathsf{g}^{\alpha\beta}\mathcal{H}_{,\alpha\beta} = \Box\mathcal{H} \equiv \gamma\rho, \tag{32}$$

where γ is the gravity constant. The latter equality holds due to the Newton's gravity law. Comparison of equation (32) with the definition of $\vec{d}$ (26) gives correspondence

$$\vec{d} \equiv \frac{1}{\gamma}\vec{\xi}. \tag{33}$$

Note that the linearity of this equation is not an intrinsic feature of the theory but is based on the exterior additional assumption, namely the Newton's gravity law with $\gamma = \text{const}$. Relation (33) lessens the number of unknowns by four. Some other closing relation will be considered in the following sections.

3.1.3. First level theory: particular case $c = \infty$. The standard fluid mechanics may be considered as the limit case of the causal fluid model when $c \to \infty$. This passage to limit as well as relations between standard three-dimensional fluid mechanics and four-dimensional non-relativistic causal theory of perfect and viscous fluids has been considered in detail in [3]. Both theories differ in a number of points and these differences are of two kinds—conceptual and qualitative—and this allows drawing parallels between two sets of notions and ideas. The attempt is made to find out correspondence between such classical notions as time, forces, potential energy, etc, and new ones including events, world lines, metrics, curvature and so on. The qualitative differences between the two models lead to such side effects as necessity of new formulations for the first and the second laws of thermodynamics. These formulations together with discussion are provided.

3.1.4. Second level theory: particular case $c = \text{const}$. For simplicity, all further considerations will be carried out for the Cartesian basis. The electromagnetic interpretation of this particular case will be given in section 3.2.

4D vorticity 2-form and 3D vorticity vector. First, consider the vorticity 2-form $\tilde{\mathsf{w}}$ and equation (22). Since $v^0 = \text{const}$, one may write

$$2\tilde{\mathsf{w}} = \begin{pmatrix} 0 & -v_{1,0} & -v_{2,0} & -v_{3,0} \\ v_{1,0} & 0 & v_{1,2}-v_{2,1} & v_{1,3}-v_{3,1} \\ v_{2,0} & v_{2,1}-v_{1,2} & 0 & v_{2,3}-v_{3,2} \\ v_{3,0} & v_{3,1}-v_{1,3} & v_{3,2}-v_{2,3} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -v_{1,0} & -v_{2,0} & -v_{3,0} \\ v_{1,0} & 0 & -\omega^3 & \omega^2 \\ v_{2,0} & \omega^3 & 0 & -\omega^1 \\ v_{3,0} & -\omega^2 & \omega^1 & 0 \end{pmatrix}.$$

Here, the definition of the 3D vorticity vector $\vec{\omega}$

$$v_{i,j} - v_{j,i} \equiv \varepsilon_{jik}\omega^k \tag{34}$$

has been used. It makes correspondence between the real components of the form $\tilde{\mathsf{w}}$ and the components of the vector $\vec{\omega}$. The quantity ε_{ijk} is a three-index Levi-Civita symbol. The exterior derivative $2\,\mathrm{d}\tilde{\mathsf{w}}$ is a zero 3-form. Indeed, the calculation shows that

$$2\,\mathrm{d}\tilde{\mathsf{w}} = -(\nabla, \vec{\omega})\,\mathrm{d}x^{123} + \left((v_{3,2} - v_{2,3})_{,0} - \omega^1_{,0}\right)\mathrm{d}x^{023}$$
$$- \left((v_{1,3} - v_{3,1})_{,0} - \omega^2_{,0}\right)\mathrm{d}x^{013} + \left((v_{2,1} - v_{1,2})_{,0} - \omega^3_{,0}\right)\mathrm{d}x^{012} = 0.$$

Hereafter, the abbreviation $\mathrm{d}x^{\alpha\beta\gamma} \equiv \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$ is used. The last three components actually coincide with time derivatives of the components of definition (34). The first one is equal to zero due to the same definition

$$(\nabla, \vec{\omega}) = 0. \tag{35}$$

4D Lamb 1-form and its 3D analogue. Consider the 3D variant of the equation of motion (24). Using definition (34), a kth component of the Lamb 1-form may be written as

$$\begin{aligned}\ell_k &= 2v^\alpha \mathsf{w}_{k\alpha} = \partial_t v_k + 2v^n \mathsf{w}_{kn} = \partial_t v_k + \varepsilon_{kni} v^n \omega^i \\ &= \partial_t v_k + (\vec{\omega} \times \vec{u})_k = \partial_t v_k + l_k.\end{aligned} \tag{36}$$

The quantity $\vec{l} \equiv \vec{\omega} \times \vec{u}$ is known as the *Lamb vector*. Hereafter, the vector $\vec{u}$ denotes three non-constant components of the velocity vector $\vec{v} \equiv (v^0, \vec{u})$. Expression (36) allows one to write the equation of motion (24) in the form

$$\ell_k \overset{(36)}{=} \partial_t v_k + l_k \overset{(24)}{=} \mathcal{H}_{,k} + f_k \quad \Rightarrow \quad \partial_t v_k + (l_k - f_k) = \mathcal{H}_{,k}$$

or

$$\partial_t \vec{u} + \vec{\epsilon} = \mathsf{g}_3^{-1}(\nabla\mathcal{H}), \tag{37}$$

where g_3 is the 3D (space) metric tensor, $\vec{\epsilon} \equiv \vec{l} - \vec{f}_3 = (\vec{\omega} \times \vec{u}) - \vec{f}_3$, $\vec{l} = \mathsf{g}_3^{-1}(\tilde{l})$, $\vec{f}_3 = \mathsf{g}_3^{-1}(\tilde{f}_3)$ and $\tilde{f} = (f_0, \tilde{f}_3)$. When $\vec{f}_3$ is equal to zero (the fluid is inviscid and incompressible), the latter equation transforms to the known *Gromeka–Lamb form of the equation of motion* of the perfect fluid

$$\partial_t \vec{u} = -\vec{l} - \mathsf{g}_3^{-1}(\nabla\mathcal{H}).$$

Induction and polarization. Since definition (26) does not describe the vector field $\vec{d}$ uniquely we may choose $\vec{d} \equiv (d^0, \vec{d}_3)$ where $d^0 = \text{const}$ and $\vec{d}_3$ is a 3D vector such that

$$\operatorname{div} \vec{d} = (\nabla, \vec{d}_3) = \rho. \tag{38}$$

Due to definition (38), the material equation (33) and the relation $\xi^n = \partial_t v^n + \epsilon^n$ (which follows from (36) and definitions of $\vec{\xi}$ and $\vec{\epsilon}$), the equalities

$$\rho = \begin{cases} d^\alpha_{,\alpha} = \dfrac{1}{\gamma}\left(\xi^0_{,0} + \xi^n_{,n}\right) = \dfrac{1}{\gamma}\left(\xi^0_{,0} + (\epsilon^n + \partial_t v^n)_{,n}\right) \\ d^n_{3,n} = \dfrac{1}{\gamma}\epsilon^n_{,n} + \dfrac{1}{\gamma}\left(\xi^0_{,0} + \partial_t v^n_{,n}\right) \end{cases}$$

hold and give the connection between two 3D vectors $\vec{d}_3$ and $\vec{\epsilon}$

$$\vec{d}_3 = \frac{1}{\gamma}\vec{\epsilon} + \vec{P} \tag{39}$$

which is valid to within an arbitrary non-divergent 3D vector. By definition $P^n_{,n} \equiv \frac{1}{\gamma}\left(\xi^0_{,0} + \partial_t v^n_{,n}\right)$. By analogy with the EMF theory vectors $\vec{d}_3$ and $\vec{P}$ may be called 3D *m-induction* and *polarization,* respectively.

Structure of the vorticity field strength. Now let us define a 2-form $\tilde{\mathsf{h}} = \mathsf{h}_{\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$ such that equation (27) holds. The exterior derivative $\mathrm{d}\tilde{\mathsf{h}}$ is as follows:

$$\begin{aligned}\mathrm{d}\tilde{\mathsf{h}} &= \tfrac{1}{2}\mathsf{h}_{\alpha\beta,\gamma}\,\mathrm{d}x^\gamma \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta\\ &= (\mathsf{h}_{01,2} + \mathsf{h}_{12,0} + \mathsf{h}_{20,1})\,\mathrm{d}x^{012} + (\mathsf{h}_{01,3} + \mathsf{h}_{13,0} + \mathsf{h}_{30,1})\,\mathrm{d}x^{013}\\ &\quad + (\mathsf{h}_{02,3} + \mathsf{h}_{23,0} + \mathsf{h}_{30,2})\,\mathrm{d}x^{023} + (\mathsf{h}_{12,3} + \mathsf{h}_{23,1} + \mathsf{h}_{31,2})\,\mathrm{d}x^{123}.\end{aligned}$$

Corresponding expression for the 3-form $*(\rho\vec{v})$ reads

$$*(\rho\vec{v}) = \rho v^0\,\mathrm{d}x^{123} - \rho v^1\,\mathrm{d}x^{023} + \rho v^2\,\mathrm{d}x^{013} - \rho v^3\,\mathrm{d}x^{012}.$$

Substituting all this to equation (27), one has

$$\rho v^0 = \mathsf{h}_{12,3} + \mathsf{h}_{23,1} + \mathsf{h}_{31,2}, \tag{40}$$

$$-\rho v^1 = (\mathsf{h}_{02,3} - \mathsf{h}_{03,2}) + \mathsf{h}_{23,0}, \tag{41}$$

$$-\rho v^2 = (\mathsf{h}_{03,1} - \mathsf{h}_{01,3}) + \mathsf{h}_{31,0}, \tag{42}$$

$$-\rho v^3 = (\mathsf{h}_{01,2} - \mathsf{h}_{02,1}) + \mathsf{h}_{12,0}. \tag{43}$$

In order to understand what all this means consider the 3D variant of the continuity equation (7). When $v^0 = \text{const}$, it takes its standard form

$$\partial_t\rho + (\nabla, \rho\vec{u}) = 0. \tag{44}$$

Using equation (38) one may write

$$(\nabla, \partial_t\vec{d}_{\mathbf{3}} + \rho\vec{u}) = 0$$

which implies the existence of the 3D vector field $\vec{\eta}$ such that

$$\partial_t\vec{d}_{\mathbf{3}} + \rho\vec{u} = \nabla \times \vec{\eta}. \tag{45}$$

Comparing equations (38) with (40) and (45) with (41)–(43), one finds correspondences

$$\begin{aligned}\mathsf{h}_{01} &= \eta^1, &\qquad \mathsf{h}_{23} &= v^0 d_{\mathbf{3}}^1,\\ \mathsf{h}_{02} &= \eta^2, & \mathsf{h}_{13} &= -v^0 d_{\mathbf{3}}^2,\\ \mathsf{h}_{03} &= \eta^3, & \mathsf{h}_{12} &= v^0 d_{\mathbf{3}}^3,\end{aligned}$$

and the matrix of the 2-form $\tilde{\mathsf{h}}$ in co-ordinate basis is as follows:

$$\tilde{\mathsf{h}} = \begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3\\ -\eta^1 & 0 & v^0 d_{\mathbf{3}}^3 & -v^0 d_{\mathbf{3}}^2\\ -\eta^2 & -v^0 d_{\mathbf{3}}^3 & 0 & v^0 d_{\mathbf{3}}^1\\ -\eta^3 & v^0 d_{\mathbf{3}}^2 & -v^0 d_{\mathbf{3}}^1 & 0 \end{pmatrix}. \tag{46}$$

4D vorticity equation and its 3D analogue. Consider the vorticity equation (28). Since the co-ordinate x^0 is imaginary, the matrix $\xi_{\alpha,\beta} - \xi_{\beta,\alpha}$ of the 2-form $\mathrm{d}\tilde{\xi}$ written in co-ordinate basis contains real $(\alpha = j, \beta = k)$ and imaginary $(\alpha = 0, \beta = k)$, $(\alpha = j, \beta = 0)$ submatrices, each of which has three independent components.

(1) First, consider real components of the 4D vorticity equation (28)

$$\begin{aligned}0 = \xi_{k,j} - \xi_{j,k} &= 2\,\underbrace{((v^0\mathsf{w}_{k0})_{,j} - (v^0\mathsf{w}_{j0})_{,k})}_{\partial_t\mathsf{w}_{kj}} + 2((v^n\mathsf{w}_{kn})_{,j} - (v^n\mathsf{w}_{jn})_{,k}) - (f_{k,j} - f_{j,k})\\ &= \varepsilon_{kji}(\partial_t\vec{\omega} + \nabla \times ((\vec{\omega} \times \vec{u}) - \vec{f}_{\mathbf{3}}))^i.\end{aligned}$$

The latter expression gives the 3D vorticity equation

$$\partial_t\vec{\omega} + \nabla \times \vec{\epsilon} = 0. \tag{47}$$

(2) Now consider imaginary components of equation (28). Taking into account equation (33), one may write

$$0 = \xi_{0,j} - \xi_{j,0} = \gamma(d_{0,j} - d_{j,0}) = \gamma(d_{0,j} - (\tilde{d}_{\mathbf{3}} + \tilde{\delta})_{j,0}), \qquad \delta_{j,0} \equiv d_{0,j} - \partial_t(d_{\mathbf{3}})_j$$

or

$$\partial_t(d_{\mathbf{3}})_j + (\delta_{j,0} - d_{0,j}) = 0.$$

Due to equation (45), the latter expression may be substituted for

$$(\partial_t \vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_j = 0.$$

Hereafter, for the sake of brevity $(\vec{a})_j$ stands for $(\mathsf{g}(\vec{a}))_j$. Thus, the imaginary components of equation (28) become as follows:

$$0 = \xi_{0,j} - \xi_{j,0} = \gamma(\partial_t \vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_j, \tag{48}$$

or using (39)

$$0 = \xi_{0,j} - \xi_{j,0} = \gamma\left(\partial_t\left(\frac{1}{\gamma}\vec{\epsilon} + \vec{P}\right) + \rho\vec{u} - \nabla \times \vec{\eta}\right)_j.$$

(3) Finally, the lower left corner of resulting antisymmetric matrix is as follows:

$$\begin{pmatrix} 0 & & \text{antisymmetric} & \\ \gamma(\partial_t \vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_1 & 0 & \text{components} & \\ \gamma(\partial_t \vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_2 & (\partial_t\vec{\omega} + \nabla \times \vec{\epsilon})_3 & 0 & \\ \gamma(\partial_t \vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_3 & -(\partial_t\vec{\omega} + \nabla \times \vec{\epsilon})_2 & (\partial_t\vec{\omega} + \nabla \times \vec{\epsilon})_1 & 0 \end{pmatrix}.$$

Current density and magnetization. Now consider time derivative of the Lamb vector $\vec{l}$. Taking into account the equation of motion (37) and the vorticity equation (47), one may write

$$\begin{aligned} \partial_t \vec{l} &= \partial_t(\vec{\omega} \times \vec{u}) = \partial_t\vec{\omega} \times \vec{u} + \vec{\omega} \times \partial_t\vec{u} \\ &= -(\nabla \times \vec{\epsilon}) \times \vec{u} - \vec{\omega} \times \left(\vec{\epsilon} - \mathsf{g}_{\mathbf{3}}^{-1}(\nabla\mathcal{H})\right). \end{aligned} \tag{49}$$

To obtain the desired form of this equation, we try to extract curl from the first term on the right side. We make use of the known relations

$$\nabla(\vec{u}, \vec{\epsilon}) = \begin{cases} \underbrace{\nabla(\vec{u}, \vec{\omega} \times \vec{u})}_{=0} - \nabla(\vec{u}, \vec{f}_{\mathbf{3}}) \\ (\vec{u}, \nabla)\vec{\epsilon} + (\vec{\epsilon}, \nabla)\vec{u} + \vec{u} \times (\nabla \times \vec{\epsilon}) + \vec{\epsilon} \times (\nabla \times \vec{u}). \end{cases}$$

This allows one to express the first term on the right side of equation (49)

$$(\nabla \times \vec{\epsilon}) \times \vec{u} = (\vec{u}, \nabla)\vec{\epsilon} + (\vec{\epsilon}, \nabla)\vec{u} + \vec{\epsilon} \times \vec{\omega} + \nabla(\vec{u}, \vec{f}_{\mathbf{3}}). \tag{50}$$

To find the expression for the first term on the right side of this relation, we consider two variants of expansion for $\nabla \times (\vec{u} \times \vec{\epsilon})$

$$\nabla \times (\vec{u} \times \vec{\epsilon}) = \begin{cases} \nabla \times (\vec{u} \times (\vec{l} - \vec{f}_{\mathbf{3}})) = \nabla \times (|\vec{u}|^2\vec{\omega} - (\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}), \\ \vec{u}(\nabla, \vec{\epsilon}) - \vec{\epsilon}(\nabla, \vec{u}) + (\vec{\epsilon}, \nabla)\vec{u} - (\vec{u}, \nabla)\vec{\epsilon}, \end{cases}$$

which give

$$(\vec{u}, \nabla)\vec{\epsilon} = -\nabla \times (|\vec{u}|^2\vec{\omega} - (\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}) + \vec{u}(\nabla, \vec{\epsilon}) - \vec{\epsilon}(\nabla, \vec{u}) + (\vec{\epsilon}, \nabla)\vec{u}.$$

Substituting this relation into equation (50), one obtains

$$\begin{aligned} -(\nabla \times \vec{\epsilon}) \times \vec{u} &= \nabla \times (|\vec{u}|^2\vec{\omega} - (\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}) \\ &\quad - \vec{u}(\nabla, \vec{\epsilon}) + \vec{\epsilon}(\nabla, \vec{u}) - 2(\vec{\epsilon}, \nabla)\vec{u} - \vec{\epsilon} \times \vec{\omega} - \nabla(\vec{u}, \vec{f}_{\mathbf{3}}). \end{aligned} \tag{51}$$

Next, we try to find the expression for the term $\vec{u}(\nabla, \vec{\epsilon})$ calculating the divergence of equation (37) and taking into account the 3D Newton's gravity law $\Delta\mathcal{H} = \gamma\rho$ which give

$$\vec{u}(\nabla, \vec{\epsilon}) = \gamma\rho\vec{u} - \vec{u}\partial_t(\nabla, \vec{u}).$$

Now (49) is as follows:

$$\partial_t\vec{l} = \nabla \times (|\vec{u}|^2\vec{\omega} - (\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}) + \big(-(\gamma\rho - \partial_t(\nabla, \vec{u}))\vec{u} \\ + \big(\vec{\epsilon}(\nabla, \vec{u}) - 2(\vec{\epsilon}, \nabla)\vec{u} - \nabla(\vec{u}, \vec{f}_{\mathbf{3}}) + \vec{\omega} \times \mathsf{g}_{\mathbf{3}}^{-1}(\nabla\mathcal{H})\big)\big). \tag{52}$$

Since $\gamma\vec{d}_{\mathbf{3}} = \vec{\epsilon} + \gamma\vec{P} = \vec{l} - \vec{f}_{\mathbf{3}} + \gamma\vec{P}$, the previous result may be rewritten as equation (45)

$$\partial_t\vec{d}_{\mathbf{3}} + \bigg(\rho\vec{u} + \frac{1}{\gamma}(-\vec{u}\partial_t(\nabla, \vec{u}) - \vec{\epsilon}(\nabla, \vec{u}) + 2(\vec{\epsilon}, \nabla)\vec{u} \\ + \nabla(\vec{u}, \vec{f}_{\mathbf{3}}) + \partial_t(\vec{f}_{\mathbf{3}} - \gamma\vec{P}) - \vec{\omega} \times \mathsf{g}_{\mathbf{3}}^{-1}(\nabla\mathcal{H}))\bigg) \\ = \frac{1}{\gamma}\nabla \times (|\vec{u}|^2\vec{\omega} - (\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}).$$

Comparing both equations, one finds correspondence

$$\vec{\eta} = \frac{|\vec{u}|^2}{\gamma}\vec{\omega} - \vec{M}, \\ \vec{M} = \frac{1}{\gamma}((\vec{u}, \vec{\omega})\vec{u} - \vec{u} \times \vec{f}_{\mathbf{3}}), \\ \vec{j} = \rho\vec{u} + \frac{1}{\gamma}\big(-\vec{u}\partial_t(\nabla, \vec{u}) - \vec{\epsilon}(\nabla, \vec{u}) + 2(\vec{\epsilon}, \nabla)\vec{u} + \nabla(\vec{u}, \vec{f}_{\mathbf{3}}) + \partial_t(\vec{f}_{\mathbf{3}} - \gamma\vec{P}) - \vec{\omega} \times \mathsf{g}_{\mathbf{3}}^{-1}(\nabla\mathcal{H})\big). \tag{53}$$

By analogy with the EMF theory, the above-introduced vectors $\vec{M}$ and $\vec{j}$ may be called the *magnetization* and *density of current*, correspondingly.

Energy balance. To obtain the 3D energy balance equation, we first calculate the dual tensor $*\tilde{\mathsf{h}}$

$$*\tilde{\mathsf{h}} = \begin{pmatrix} 0 & v^0 d_{\mathbf{3}}^1 & v^0 d_{\mathbf{3}}^2 & v^0 d_{\mathbf{3}}^3 \\ -v^0 d_{\mathbf{3}}^1 & 0 & \eta^3 & -\eta^2 \\ -v^0 d_{\mathbf{3}}^2 & -\eta^3 & 0 & \eta^1 \\ -v^0 d_{\mathbf{3}}^3 & \eta^2 & -\eta^1 & 0 \end{pmatrix} \tag{54}$$

and use expressions (47) and (48). Now the contraction $\mathrm{d}\xi(*\tilde{\mathsf{h}})$ reads

$$\mathrm{d}\xi(*\tilde{\mathsf{h}}) = 2\big(v^0 d_{\mathbf{3}}^j\gamma(\partial_t\vec{d}_{\mathbf{3}} + \rho\vec{u} - \nabla \times \vec{\eta})_j + \eta^j(\partial_t\vec{\omega} + \nabla \times \vec{\epsilon})_j\big) = 0 \tag{55}$$

and is equal to zero in accordance with the energy conservation law (30). In a particular case when $\vec{P} = 0$ and $\gamma\vec{d}_{\mathbf{3}} = \vec{\epsilon}$, the energy conservation law (55) simplifies to

$$(v^0(\vec{d}_{\mathbf{3}}, \partial_t\vec{\epsilon}) + (\vec{\eta}, \partial_t\vec{\omega})) + v^0(\vec{\epsilon}, \rho\vec{u}) + (\nabla, \vec{\epsilon} \times \vec{\eta}) = 0. \tag{56}$$

3D system of the second level equations. Finally, let us list the equations of the resulting 3D system. They are equations (38), (45) and (47) together with closing relations (39) and (53). Thirteen scalar equations bind 16 unknowns $(\rho, \vec{u}, \vec{d}_{\mathbf{3}}, \vec{\eta}, \vec{\omega}, \vec{\epsilon})$. Hence, one more 3D vector relation is required. In the EMF theory, such relation is known as the Ohm's law.

3.2. Electromagnetic interpretation: Maxwell's equations

Electromagnetic interpretation of the particular case (section 3.1.4) of equations (26)–(28) is demonstrated.

(1) Conserved measure will be denoted here by q and called the *electric charge*[1]. Its density will be denoted by ρ_e (cf (6))

$$q(\mathcal{B}_t) = \int_{V(\mathcal{B}_t)} \rho_e \,\mathrm{d}V.$$

(2) The electric charge density induces the *electric induction* vector field $\vec{D}$ such that

$$(\nabla, \vec{D}) \equiv \rho_e. \tag{57}$$

This expression is known as the *Gauss Law for* $\vec{D}$ (cf equation (38)).

(3) The conservation of q induces the continuity equation for the electric charges (cf equation (44))

$$\partial_t \rho_e + (\nabla, \rho_e \vec{u}) = 0. \tag{58}$$

It follows from the 4D continuity equation in the case $v^0 = \text{const}$. In the 3D case, equation (27) may be written in the usual form of the *Ampère's law*. Indeed, using (57), equation (58) reads

$$(\nabla, \partial_t \vec{D} + \rho_e \vec{u}) = 0$$

which implies the Ampère's law since $(\nabla, \nabla \times \vec{H}) = 0$ for arbitrary vector field $\vec{H}$ (cf equation (45))

$$\partial_t \vec{D} + \rho_e \vec{u} = \nabla \times \vec{H}. \tag{59}$$

The vector field $\vec{H}$ is called the *magnetic field strength*. It is clear that the tensor $*\tilde{\mathsf{h}}$ in expression (54) corresponds to the known Faraday tensor F (see, e.g., [14]):

$$\mathsf{F} = \begin{pmatrix} 0 & v^0 D^1 & v^0 D^2 & v^0 D^3 \\ -v^0 D^1 & 0 & H^3 & -H^2 \\ -v^0 D^2 & -H^3 & 0 & H^1 \\ -v^0 D^3 & H^2 & -H^1 & 0 \end{pmatrix}.$$

(4) Using the notation $\vec{B} \equiv \vec{\omega}$, equation (35) turns to the *Gauss' law for* $\vec{B}$

$$(\nabla, \vec{B}) \equiv 0, \tag{60}$$

where $\vec{B}$ is called the *magnetic induction*.

(5) The last *Faraday's law* of the EMF theory is just equation (47) written in the new notation ($\vec{B} \equiv \vec{\omega}$, $\vec{E} \equiv \vec{\epsilon}$)

$$\partial_t \vec{B} + \nabla \times \vec{E} = 0. \tag{61}$$

Here, $\vec{E}$ is the *electric field strength*.

Thus, all the equations of the Maxwell's theory are present. Namely, equations (57), (59)–(61) make the system of equations of the EMF theory. They correspond to equations (38), (45), (35) and (47) of the particular case of the ACM. The energy conservation law (55) and

[1] Although physical charges may be of different signs whereas the mathematical measure is positive by definition, there is no contradiction here, since the charges with different signs are not considered in the current research. In case of need, the measure may be substituted for the mathematical notion of charge (see, e.g., [15]).

Table 2. Parallels between ACM, 3D fluid mechanics and EMF definitions. Here, $\vec{v} = (v^0, \vec{u})$, $\vec{d} = (d^0, \vec{d}_3)$, $\vec{\xi} = (\xi^0, \vec{\epsilon})$, $\vec{\ell} = (\ell^0, \partial_t \vec{v} + \vec{l})$, $\vec{f} = (f^0, \vec{f}_3)$.

	ACM notions	4D hydromechanics	3D hydromechanics	Equivalent EMF notions
1	Measure	Mass m		Electric charge q
2	Density	Mass density ρ: $\int_V \rho \, dV = m$		Electric charge density ρ_e: $\int_V \rho_e \, dV = q$
3	Induction	$\langle$Gravitational induction$\rangle$ $\vec{d}$: $\operatorname{div} \vec{d} = \rho$	$(\nabla, \vec{d}_3) = \rho$	Electric induction $\vec{D}$ (Gauss' law for $\vec{D}$): $(\nabla, \vec{D}) = \rho_e$
4	Tangent vector	Velocity vector $\vec{v}$	$\vec{u}$	Vector potential $\vec{A}$
5	Conservation of measure	Mass conservation: $d_t m = 0$ Continuity equation: $\operatorname{div} \rho\vec{v} = 0$	$\partial_t \rho + (\nabla, \rho\vec{u}) = 0$	Electric charge conservation: $d_t q = 0$ Continuity equation for q: $\partial_t \rho_e + (\nabla, \rho_e \vec{u}) = 0$
6	Strength of the density field	$\langle$Vorticity field strength$\rangle$ $\tilde{\mathrm{h}}$: $d\tilde{\mathrm{h}} = *\rho\vec{v}$	$\partial_t \vec{d}_3 + \rho\vec{u} = \nabla \times \vec{\eta}$	Magnetic field strength $\vec{H}$ (Ampère's law): $\partial_t \vec{D} + \rho\vec{u} = \nabla \times \vec{H}$
7	Exterior derivative of the field $\vec{v}$	Vorticity: $\tilde{\mathrm{w}} \equiv d\tilde{v}$ $d\tilde{\mathrm{w}} = 0$	Curl: $\vec{\omega} \equiv \nabla \times \vec{u}$ $(\nabla, \vec{\omega}) = 0$	Magnetic induction $\vec{B}$ (Gauss' law for $\vec{B}$): $(\nabla, \vec{B}) = 0,\ \vec{B} \equiv \nabla \times \vec{A}$
8	Contraction $\tilde{\ell} \equiv 2\tilde{\mathrm{w}}(\vec{v})$	Lamb 1-form: $\tilde{\ell} \equiv 2\tilde{\mathrm{w}}(\vec{v})$	Lamb vector: $\vec{l} \equiv \vec{\omega} \times \vec{u}$	
9		Bernoulli integral: $\mathcal{H}$		Electric potential: φ
10	Representation of $\tilde{\ell}$	Equation of motion: $\tilde{\ell} = d\mathcal{H} + \tilde{f}$ $\tilde{\xi} \equiv \tilde{\ell} - \tilde{f}$	$\partial_t \vec{u} + \vec{l} = \mathrm{g}_3^{-1}(\nabla\mathcal{H}) + \vec{f}_3$ $\vec{\epsilon} \equiv \vec{l} - \vec{f}_3$	Electric field strength $\vec{E}$: $\partial_t \vec{A} + \vec{E} = -\nabla\varphi$
11	Closure of $\tilde{\xi}$	Vorticity equation: $d\tilde{\xi} = 0$	$\partial_t \vec{\omega} + \nabla \times \vec{\epsilon} = 0$	Faraday's law: $\partial_t \vec{B} + \nabla \times \vec{E} = 0$
12		Newton's gravity: $\Box\mathcal{H} = \gamma\rho, \vec{\xi} = \gamma\vec{d}$	$\Delta\mathcal{H} = \gamma\rho$ $\vec{d}_3 = \frac{1}{\gamma}\vec{\epsilon} + \vec{P}$	1st material equation: $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$
			$\partial_t \vec{l} = \nabla \times \vec{\eta} - \vec{\jmath}$ $\vec{\eta} = \frac{\lvert\vec{u}\rvert^2}{\gamma}\vec{\omega} - \vec{M}$	2nd material equation: $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

particularly equation (56) correspond to the EMF energy balance equation in the form of the known Poynting equation

$$\tfrac{1}{2}((\vec{D}, \partial_t \vec{E}) + (\vec{H}, \partial_t \vec{B})) + (\nabla, \vec{E} \times \vec{H}) + (\vec{\jmath}, \vec{E}) = 0.$$

The material equations (39) and (53) correspond to the well-known EMF relations $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$, where ε_0 is permittivity, μ_0 is permeability and $\vec{P}$ and $\vec{M}$ either are known or require additional definitions. The last material equation which close the system is the Ohm's law $\sigma\vec{E} = \vec{\jmath}$, where σ is conductivity.

4. Concluding remarks

Two levels of the causal abstract continuum mechanics have been suggested and two possible interpretations of the developed theory have been considered. Besides, some particular cases which bind current investigation with known classical theories have been studied. Due to historical reasons different approaches and methods of description have been developed within the framework of both physical theories. However, in either cases all the considerations are based on measures (gravitational measure or mass, and electrical measure or charge) and identical fundamental principles which state the conservation of measure in the absence of cause. Beside these statements, both theories include postulates (the equation of motion (17) and the Faraday's law (28)) which are mathematically connected with the vorticity equation. They follow from each other in the case of sufficiently smooth functions. The above-mentioned features allow development of the general mathematical theory which incorporates both classical theories as particular cases. We tried to demonstrate such possibility. Some concluding remarks may be found in what follows.

(1) The main motive of constructing causal physical theories consists in desire to avoid contradictions with the causality principle. Straightforward way to achieve this goal is to extend an object of theoretical description beyond the phenomenon itself to include an observer and observational process and, thus, to take into account the finiteness of the velocity of observation (light, sound or whatever else).

Actually this is done via the interpretation of the synchronization method (see section 2.1.1) in terms of the velocity of observation, i.e. the signal speed. We call the signal a radiation which weakly interacts with physical continuum in study and does not change its physical properties somehow noticeably (see [3] for discussion of this topic).

What are the benefits of such an approach? Most important is that the velocity of observation appears to be an upper limit of resolved velocities and thus guarantees the causal stipulation of events. This limit, however, is not absolute, but a relative one, which makes sense within the current observation (or model, since model now includes observation). Greater velocities (if any) are interpreted erroneously by the observer. A motion with greater velocity is observed (if possible) as a motion with apparent velocity, which is smaller than the signal one (see [2, 3] for detailed consideration and simple example). The lower the signal speed, the lesser details are described by the model.

Two models of an object which differ in a signal velocity used give different results since real observations differ either. For instance, real fluids usually possess at least two signals which may be used for soundings, i.e. light and sound. It makes sense to put one of these velocities at the basis of the model which will be used to describe (or be verified by) corresponding measurements. It should be kept in mind that the model itself may forbid some interpretations of signal speed. Thus, the causal model of compressible fluid allows both interpretations of signal speed, whereas the causal model of incompressible fluid permits only light signals since the speed of sound becomes infinite within this model.

(2) First level of the ACM requires a nonzero measure (mass, charge, etc). Second level theory does not depend on this requirement and thus is able to describe distant interaction of the bodies with measures.

(3) The Maxwell's equations are known to be Lorentz invariant and this usually implicitly means that the theory is relativistic. On the contrary, the abstract continuum mechanics is declared to be non-relativistic. However, the contradiction here is seeming. It should be taken into account that the Maxwell's equations are compared with particular case of the ACM with constant signal speed. In this case, both theories permit the Lorentz

transformations (the general ACM theory permits infinitesimal Lorentz transformations). In either theory there exists upper velocity limit $c_{\max}$. In the EMF theory this is the speed of light, whereas in the ACM this is the velocity of a signal used for observation (see the detailed consideration of this topic in [2]. As has been said, it may be shown that all the velocities which exceed the velocity of the signal used for observations $c_{\max}$ cannot be truly estimated and either are unobservable or are interpreted erroneously such that their apparent values are lesser than $c_{\max}$). Besides, the Lorentz transformation does not depend on the definite value of the constant c. Moreover, due to the existence of the upper limit velocity $c_{\max}$ the ACM, being non-relativistic in the usual sense, possesses at the same time some features which are common to the relativistic theories. Thus, it allows description of the Fizeau experiment and gives the known relation $1-\frac{1}{n^2}$ for the Fresnel dragging coefficient (see [16]). It permits the conformal local transformation group just like the Maxwell's system of equations does and so forth.

(4) Introduction of the potential energy is quite simple. A part of the total energy is taken into account as the so-called *potential energy* $U(\mathcal{B}_t)$ with potential Φ and density $\rho\Phi$. In this case,

$$Cm(\mathcal{B}_t) = \int_{V(\mathcal{B}_t)} (\kappa + \rho\Phi)\,\mathrm{d}V.$$

Since the consideration of the potential energy affects the signal velocity only (see [3] for details), it always may be taken into account implicitly via the redefinition of the value of c. In the thermodynamic case, equation (8) turns into the total energy $Cm = K + U + E$ conservation law

$$d_t(K+U+E) = d_t\int_V (k+\rho\Phi+e)\,\mathrm{d}V = 0$$

and

$$\operatorname{div}(k+\rho\Phi+e)\vec{v} = \rho d_t\left(\frac{k+e}{\rho}+\Phi\right) = 0. \tag{62}$$

Equation (62) may be rewritten as follows:

$$(\operatorname{div}\mathsf{M})(\vec{v}) + \rho d_t\left(\Phi+\frac{e}{\rho}\right) = 0. \tag{63}$$

The equation of motion

$$\operatorname{div}\mathsf{M} = \operatorname{div}\mathsf{T} - \rho\,\mathrm{d}\Phi \tag{64}$$

is postulated. Taking into account the continuity equation (7), equation (64) may also be written as follows:

$$d_t\tilde{v} = \frac{1}{\rho}\operatorname{div}\mathsf{T} - \mathrm{d}\Phi. \tag{65}$$

The Bernoulli integral now reads

$$\mathcal{H} = \frac{\pi}{\rho} - \left(\frac{k}{\rho}+\Phi\right) + \nu\operatorname{div}\vec{v}.$$

(5) Beside the hydromechanical interpretation of the ACM suggested, it is possible to consider some others using different definitions of the stress tensor. The theory of elasticity gives an example. According to the Hooke's law, the stress tensor T is set to be linearly dependent on the strain tensor S (see, e.g., [4, 17]). In general case, such dependence is given by the four-rank tensor Y called the elastic modulus tensor

$$\mathsf{T} = \mathsf{Y}(\mathsf{S}).$$

Within the current context, these tensors must be considered as four dimensional with the same symmetry properties as their 3D analogues.

(6) All mentioned parallels between the sets of notions are presented in table 2. Names of two variables in this table do not exist in the scientific literature and have been invented to underline the analogy with other variables. They are enclosed in angle brackets.

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